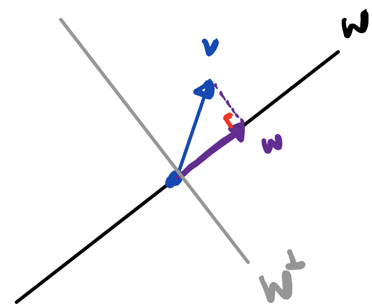
MATH 2040 Lecture 18 (10/11/2016)

Spectral Thm: T self-adjoint ($\mathbb{F}=\mathbb{R}$) or normal ($\mathbb{F}=\mathbb{C}$)

$\iff \exists$ orthonormal eigenbasis

Question: How to understand it geometrically?

Recall: $W \subseteq V$ subspace



$$\Rightarrow \text{decompose} \quad \begin{array}{ccccc} V & = & W & \oplus & W^\perp \\ \Downarrow & & \Downarrow & & \Downarrow \\ v & = & w & + & w^\perp \end{array}$$

Define $T = \mathrm{proj}_W^\perp : V \longrightarrow V$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ v & \longmapsto & w \end{array}$$

Note: (1) T linear.

(2) $T^2 = T$

(3) $R(T) = W$, $N(T) = W^\perp$ $(= R(T)^\perp)$

(4) $T^* = T$

Reason: $V = W \oplus W^\perp$

O.N. basis $\beta = \gamma_1 \cup \gamma_2$

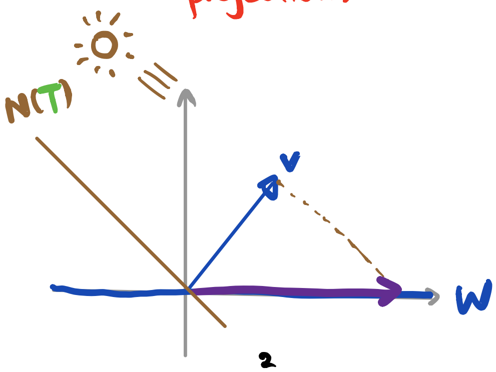
$$[T]_\beta = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{self-adjoint}$$

Def$^{\underline{n}}$: A linear map $T: V \to V$ is orthogonal projection

if ① $T^2 = T$ & ② $R(T)^\perp = N(T)$

(projection)

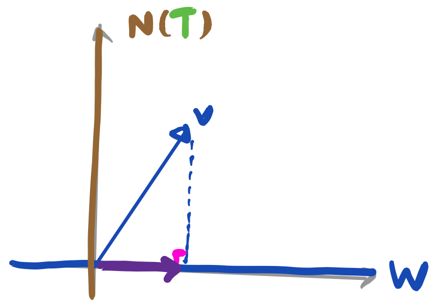
projection:



$T^2 = T$

VS

$\perp$ projection:



$T^2 = T$

& $N(T) \perp W$

Length-min. property: $\|Tv\| \leq \|v\|$ if T is $\perp$ projection.

Prop: (1) T orthogonal projection

$\Leftrightarrow$ (2) $T^2 = T = T^*$

$\Leftrightarrow$ (3) $\exists\, W \subset V$ subspace s.t. $T = \mathrm{proj}_W^{\perp}$.

Proof: (3) $\Rightarrow$ (2) done!

(2) $\Rightarrow$ (1) $R(T)^{\perp} = N(T^*) \underset{\text{adj.}}{\overset{\text{self}}{=}} N(T)$

(1) $\Rightarrow$ (3) Know: $T^2 = T^{(*)}$ & $R(T)^{\perp} = N(T).^{(*)}$

Define $W = R(T) \subseteq V$ subspace.

Check: $T = \mathrm{proj}_W^{\perp}$.

$$V = R(T) \oplus \underset{(*)}{R(T)^{\perp}}$$

$$= W \oplus N(T)$$

$$T\left(v = Tu + y\right) \Rightarrow Tv = \overset{\overset{Tu}{\overset{\parallel\,(*)}{}}}{T^2 u} + 0$$

$\square$

T diagonalizable $\iff$ $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$

T diagonalizable by an O.N.B. $\iff$ $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ and $\perp$ $\perp$

Spectral Decomposition Thm:

Assume $T: V \to V$ normal$^{(\mathbb{C})}$ / self-adjoint$^{(\mathbb{R})}$.

If T has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then

$$\boxed{V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}} \quad \& \quad E_{\lambda_i} \perp E_{\lambda_j}$$

and $$\boxed{T = \lambda_1 T_1 + \cdots + \lambda_k T_k} \qquad T_i = \mathrm{proj}^{\perp}_{E_{\lambda_i}}$$

$\cong$

matrix: β O.N.B.

$$[T]_\beta = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \lambda_k & & \\ & \ddots & \\ & & \lambda_k \end{matrix}} \end{pmatrix}$$

$$\boxed{V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}}$$

$$Tv = Tv_1 + \cdots + Tv_k = \lambda_1 v_1 + \cdots + \lambda_k v_k$$

$$\boxed{T = \lambda_1 T_1 + \cdots + \lambda_k T_k}$$

Applications:

Cor 1: $\boxed{g(T) = g(\lambda_1)T_1 + \cdots + g(\lambda_k)T_k}$ Ex.

for any polynomial $g(x)$.

Eg.: $T = \lambda_1 T_1 + \lambda_2 T_2 \Rightarrow T^2 + 2T = (\lambda_1^2 + 2\lambda_1)T_1$
$+ (\lambda_2^2 + 2\lambda_2)T_2$

$$[T]_\beta = \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} \Rightarrow [T^k]_\beta = \begin{pmatrix} \lambda_1^k I & 0 \\ 0 & \lambda_2^k I \end{pmatrix}$$

Cor 2: $\mathbb{F} = \mathbb{C}$. T normal $\Leftrightarrow T^* = g(T)$ for some polynomial g.

Pf: ($\Leftarrow$) easy. $T^*T = TT^*$ ✓

($\Rightarrow$) T normal

$$\Rightarrow T = \lambda_1 T_1 + \cdots + \lambda_k T_k \quad (*)$$

$$\Rightarrow T^* = (\lambda_1 T_1 + \cdots + \lambda_k T_k)^*$$

$$= \overline{\lambda_1} T_1^* + \cdots + \overline{\lambda_k} T_k^*$$

$$= \overline{\lambda_1} T_1 + \cdots + \overline{\lambda_k} T_k$$

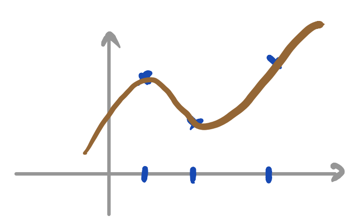
$$\overset{?}{=} g(\lambda_1) T_1 + \cdots + g(\lambda_k) T_k$$

$$= g(T)$$

□

Q: $\exists$ polynomial g s.t. $g(\lambda_i) = \overline{\lambda_i}$ $\forall i$.

A: Yes, if we allow large degree.



"Lagrange Interpolation".